More on One-to-one Functions and Onto Functions and One-to-one and Onto Functions

Theorem (NIB) 9 (Solutions to Sec 7.3, #18 and #19):

Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then, $g \circ f: X \to Z$.

1) If $g \circ f$ is one-to-one, then f is one-to-one. (f is applied *first*.)

2) If $g \circ f$ is onto, then g is onto. (g is applied *last*.)

Proof: (by contraposition)

1) Suppose that f is not one-to-one. [NTS : $g \circ f$ is not one-to-one.]
$\therefore$ There exist elements $u \in X$ and $v \in X$ are such that
$f(u) = f(v)$ and $u \neq v$. $\therefore g(f(u)) = g(f(v))$ and $u \neq v$.
$\therefore (g \circ f)(u) = (g \circ f)(v)$ and $u \neq v$. $\therefore (g \circ f)$ is not one-to-one.
$\therefore$ If $(g \circ f)$ is one-to-one, then f is one-to-one.

2) Suppose that g is not onto. [NTS: $g \circ f$ is not onto.]
$\therefore$ There exists an element $z_0 \in Z$ such that $g(y) \neq z_0$, for all $y \in Y$.
Suppose x is any element of X. Let $y_0 = f(x) \in Y$.
$\therefore g(y_0) \neq z_0$. $\therefore g(f(x)) \neq z_0$. $\therefore (g \circ f)(x) \neq z_0$.
$\therefore \forall x \in X$, $(g \circ f)(x) \neq z_0$. $\therefore (g \circ f)$ is not onto.
$\therefore$ If $(g \circ f)$ is onto, then g is onto.
QED

Theorem (NIB) 10: Let $f: X \to Y$ and $g: Y \to X$ be functions.

If $g \circ f = i_X$ and $f \circ g = i_Y$,

then f is a one-to-one correspondence and $g = f^{-1}$.

Proof: Recall: $i_X(x) = x, \forall x \in X$, and $i_Y(y) = y, \forall y \in Y$.

Since i_X is one-to-one and since f is applied first in $g \circ f$,
f is one-to-one by Part 1) of Theorem (NIB) 4.
Since i_Y is onto and since f is applied last in $f \circ g$,
f is onto Part 2) of Theorem (NIB) 4.
$\therefore f$ is a one-to-one correspondence.

The proof that $g = f^{-1}$ is left as an exercise. (#25 of Sec. 7.3) Q E D

"The Test for a One-to-one Correspondence":

Theorem (NIB) 10 can be used to prove that a function f is a one-to-one correspondence by taking the following steps:

Step 1) Compute what the formula for $g = f^{-1}$ should be.

Step 2) Show that $g \circ f = i_X$ and $f \circ g = i_Y$; that is, show that $g(f(x)) = x$, for all $x \in X$, and that $f(g(y)) = y$, for all $y \in Y$.

Example Problem:

Using the "Test for a One-to-one Correspondence", prove that f is a one-to-one correspondence from $\mathbb{R}$ to $\mathbb{R}^+$, where:

For all $x \in \mathbb{R}$, $f(x) = 5e^{3x} \in \mathbb{R}^+$.

Solution: Recall that, $\forall\, x \in \mathbb{R}$, $e^{\ln x} = x$, and $\forall\, y \in \mathbb{R}^+$, $\ln\left(e^{y}\right) = y$.

Step 1) Compute what the formula for $g = f^{-1}$ should be:

$$g(y) = x \Leftrightarrow f^{-1}(y) = x \Leftrightarrow f(x) = y \Leftrightarrow 5e^{3x} = y \Leftrightarrow e^{3x} = y/5$$

$$\Leftrightarrow 3x = \ln\left(\frac{y}{5}\right) \Leftrightarrow x = \frac{1}{3}\ln\left(\frac{y}{5}\right)$$

Define function $g: \mathbb{R}^+ \to \mathbb{R}$ as follows: For all $y \in \mathbb{R}^+$, $g(y) = \frac{1}{3}\ln\left(\frac{y}{5}\right)$.

Step 2) Show that $g \circ f = i_X$ and $f \circ g = i_Y$; that is, show that $g(f(x)) = x$, for all $x \in X$, and that $f(g(y)) = y$, for all $y \in Y$.

$$f(g(y)) = f\left(\frac{1}{3}\ln\left(\frac{y}{5}\right)\right) = 5e^{\left(3\left(\frac{1}{3}\ln\left(\frac{y}{5}\right)\right)\right)} = 5e^{\left(\ln\left(\frac{y}{5}\right)\right)} = 5\,\frac{y}{5} = y$$

$$g(f(x)) = g\left(5e^{3x}\right) = \frac{1}{3}\ln\left(\frac{5e^{3x}}{5}\right) = \frac{1}{3}\ln\left(e^{3x}\right) = \frac{1}{3}\,3x = x$$

Therefore, f is a one-to-one correspondence and $g = f^{-1}$ by Theorem (NIB) 10.